Circular Flow in Mono-directed Eulerian Signed Graphs

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2 Circular flow in mono-directed Eulerian signed graphs
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Jaeger's circular flow conjecture

Tutte's 3-flow conjecture

Every 4-edge-connected graph admits a nowhere-zero 3-flow.

Jaeger's circular flow conjecture

Every $4k$-edge-connected graph admits a circular $\frac{2k+1}{k}$-flow.

- It has been disproved for $k \geq 3$ [M. Han, J. Li, Y. Wu, and C.Q. Zhang 2018];
- It has been verified that the $6k$-edge-connectivity is a sufficient condition for a graph to admit a circular $\frac{2k+1}{k}$-flow [L. M. Lovász, C. Thomassen, Y. Wu, and C.Q. Zhang 2013].

Duality: circular flow and circular coloring

For any positive integers p, q with $p \geq 2q$, a circular $\frac{p}{q}$-flow in a graph G is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}$ satisfying that $q \leq |f(e)| \leq p - q$ and for each vertex v, $\sum_{(v,w)\in D} f(vw) - \sum_{(u,v)\in D} f(uv) = 0$.

For any positive integers p, q with $p \geq 2q$, a circular $\frac{p}{q}$-coloring of a graph G is a mapping $\varphi : V(G) \to \{1, 2, \ldots, p\}$ such that $q \leq |f(u) - f(v)| \leq p - q$ for each edge $uv \in E(G)$.

Lemma [L. A. Goddyn, M. Tarsi, and C.Q. Zhang 1998]

A plane graph G admits a circular $\frac{p}{q}$-coloring if and only if its dual graph G^* admits a circular $\frac{p}{q}$-flow.

Jaeger-Zhang Conjecture

Jaeger-Zhang Conjecture [C.-Q. Zhang 2002]

Every planar graph of odd-girth at least $4k+1$ admits a circular $\frac{2k+1}{k}$-coloring.

- $k = 1$: Grötzsch's theorem;
- $k = 2$: true for odd-girth 11 [Z. Dvořák and L. Postle 2017; D. Cranston and J. Li 2020];
- $k = 3$; true for odd-girth 17 [D. Cranston and J. Li 2020; L. Postle and E. Smith-Roberge 2022];
- $k \geq 4$:
  - true for odd-girth $8k-3$ [X. Zhu 2001];
  - true for odd-girth $\frac{20k-2}{3}$ [O.V. Borodin, S.-J. Kim, A.V. Kostochka and D.B. West 2002];
  - true for odd-girth $6k+1$ [L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013].

Signed graphs

- A signed graph (G, σ) is a graph G together with an assignment $\sigma : E(G) \to \{+, -\}$.
- The sign of a closed walk (especially, a cycle) is the product of signs of all the edges in it.
- A switching at vertex v is to switch the signs of all the edges incident to this vertex.

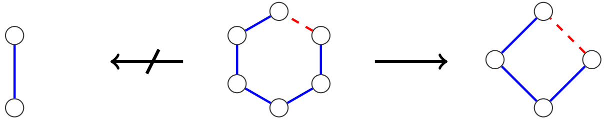
Theorem [T. Zaslavsky 1982]

Signed graphs (G, σ) and (G, σ') are switching equivalent if and only if they have the same set of negative cycles.

- The negative-girth of a signed graph is the length of a shortest negative cycle.

Homomorphism of signed graphs

- A homomorphism of (G, σ) to (H, π) is a mapping φ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ respectively, such that the adjacency, the incidence and the signs of closed walks are preserved.
- A homomorphism of (G, σ) to (H, π) is said to be edge-sign preserving if furthermore, it preserves the signs of the edges.
- $(G, \sigma) \to (H, \pi) \Leftrightarrow \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi)$.



Circular coloring of signed graphs

Let C^r be a circle of circumference r.

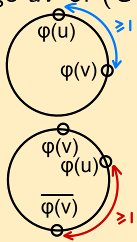
Definition [R. Naserasr, Z. Wang and X. Zhu 2021]

Given a signed graph (G, σ) with no positive loop and a real number r, a circular r-coloring of (G, σ) is a mapping $\varphi : V(G) \to C^r$ such that for each positive edge uv of (G, σ),

$$d_{C^r}(\varphi(u), \varphi(v)) \geq 1,$$

and for each negative edge uv of (G, σ),

$$d_{C^r}(\varphi(u), \overline{\varphi(v)}) \geq 1.$$



The circular chromatic number of (G, σ) is defined as

$$\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.$$

Circular $\frac{p}{q}$-coloring of signed graphs

For $i, j, x \in \{0, 1, \ldots, p-1\}$,

$$d_{(\text{mod } p)}(i, j) = \min\{|i-j|, p-|i-j|\} \text{ and } \bar{x} = x + \frac{p}{2} (\text{mod } p).$$

Given a positive even integer p and a positive integer q satisfying $q \leq \frac{p}{2}$, a circular $\frac{p}{q}$-coloring of a signed graph (G, σ) is a mapping $\varphi : V(G) \to \{0, 1, \ldots, p-1\}$ such that for any positive edge uv,

$$q \leq |\varphi(u) - \varphi(v)| \leq p - q,$$

and for any negative edge uv,

$$|\varphi(u) - \varphi(v)| \leq \frac{p}{2} - q \quad \text{or} \quad |\varphi(u) - \varphi(v)| \geq \frac{p}{2} + q.$$

Orientation on signed graphs

- A signed graph is bi-directed if each edge is assigned with two directions at both of its ends such that
  - in a positive edge, the ends are both directed from one endpoint to the other,
  - in a negative edge, either both ends are directed outward or both are directed inward.
- A signed graph is mono-directed if each edge is assigned with one direction.

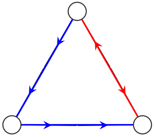


Figure: A bi-directed signed K_3

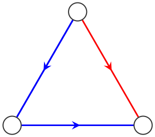


Figure: A mono-directed signed K_3

Circular $\frac{p}{q}$-flow in mono-directed signed graphs

Definition [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Given a positive even integer p and a positive integer q where $q \leq \frac{p}{2}$, a circular $\frac{p}{q}$-flow in a signed graph (G, σ) is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}$ satisfies the followings.

- For each positive edge e, $|f(e)| \in \{q, \ldots, p-q\}$.
- For each negative edge e, $|f(e)| \in \{0, \ldots, \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, \ldots, p-1\}$.
- For each vertex v of (G, σ), $\sum_{(v,w)\in D} f(vw) = \sum_{(u,v)\in D} f(uv)$.

The circular flow index of (G, σ) is defined to be

$$\Phi_c(G, \sigma) = \min\{\frac{p}{q} \mid (G, \sigma) \text{ admits a circular } \frac{p}{q}\text{-flow}\}.$$

Circular $\frac{2\ell}{\ell-1}$-flow and circular $\frac{2\ell}{\ell-1}$-coloring

Let k be a positive integer.

- A signed graph $(G,+)$ admits a circular $\frac{2k+1}{k}$-coloring if and only if $(G,+) \to C_{2k+1}$.
- A signed bipartite graph (G,σ) admits a circular $\frac{4k}{2k-1}$-coloring if and only if $(G,\sigma) \to C_{-2k}$. [R. Naserasr and Z. Wang 2021]

Lemma [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

A signed plane graph (G,σ) admits a circular $\frac{p}{q}$-coloring if and only if its dual signed graph (G^*,σ^*) admits a circular $\frac{p}{q}$-flow, i.e.,

$$\chi_c(G,\sigma) \leq \frac{p}{q} \quad \Leftrightarrow \quad \Phi_c(G^*,\sigma^*) \leq \frac{p}{q}.$$

Example

$$\chi_c(G,\sigma) \le \frac{p}{q} \quad \Leftrightarrow \quad \Phi_c(G^*,\sigma^*) \le \frac{p}{q}.$$

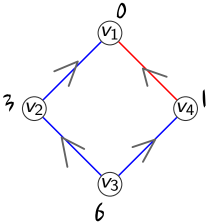


Figure: Circular $\frac{8}{3}$-coloring of C_{-4}

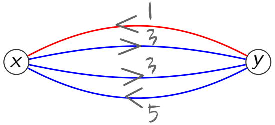


Figure: Circular $\frac{8}{3}$-flow in C^*_{-4}

Signed bipartite analog of Jaeger-Zhang conjecture

Signed bipartite analog of Jaeger's circular flow conjecture

Every $g(k)$-edge-connected Eulerian signed graph admits a circular $\frac{4k}{2k-1}$-flow.

Signed bipartite analog of Jaeger-Zhang conjecture

Every signed bipartite planar graph of negative-girth at least $f(k)$ admits a homomorphism to C_{-2k}.

- It was conjectured that $f(k) = 4k - 2$ [R. Naserasr, E. Rollová, and É. Sopena 2015];
- However, for $k = 2$, 8 is proved to be the best negative-girth condition [R. Naserasr, L-A. Pham, and Z. Wang 2022];
- For any $k \geq 3$, true for negative-girth $8k - 2$ [C. Charpentier, R. Naserasr, and E. Sopena 2020].

Main results

Theorem [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Every $(6k-2)$-edge-connected Eulerian signed graph admits a circular $\frac{4k}{2k-1}$-flow.

Theorem [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Every signed bipartite planar graph of negative-girth at least $6k-2$ admits a homomorphism to C_{-2k}.

$(2k, \beta)$-orientation on graphs

Definition [J. Li, Y. Wu and C.Q. Zhang 2020]

Given a graph G, a function $\beta : V(G) \to \{0, \pm 1, \ldots, \pm k\}$ is a $(2k, \beta)$-boundary of G if for every vertex $v \in V(G)$,

$$\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{2k} \quad \text{and} \quad \beta(v) \equiv d(v) \pmod{2}.$$

Given a subset $A \subset V(G)$, we define $\beta(A) \in \{0, \pm 1, \ldots, \pm k\}$ such that $\beta(A) \equiv \sum_{v \in A} \beta(v) \pmod{2k}$.

Given a $(2k, \beta)$-boundary β, an orientation D on G is called a $(2k, \beta)$-orientation if for every vertex $v \in V(G)$,

$$\overleftarrow{d_D}(v) - \overrightarrow{d_D}(v) \equiv \beta(v) \pmod{2k}.$$

$(2k, \beta)$-orientation on graphs

Theorem [L.M. Lovasz, C. Thomassen, Y. Wu and C.Q. Zhang 2013; J. Li, Y. Wu and C.Q. Zhang 2020]

Let G be a graph with a $(2k, \beta)$-boundary β for $k \geq 3$. Let z_0 be a vertex of $V(G)$ such that $d(z_0) \leq 2k - 2 + |\beta(z_0)|$. Assume that D_{z_0} is an orientation on $E(z_0)$ which achieves the boundary $\beta(z_0)$. Let $V_0 = \{v \in V(G) \setminus \{z_0\} \mid \beta(v) = 0\}$. If $V_0 \neq \emptyset$, we let v_0 be a vertex of V_0 with the smallest degree. Assume that $d(A) \geq 2k - 2 + |\beta(A)|$ for any $A \subset V(G) \setminus \{z_0\}$ with $A \neq \{v_0\}$ and $|V(G) \setminus A| > 1$. Then the partial orientation D_{z_0} can be extended to a $(2k, \beta)$-orientation on the entire graph G.

Theorem [J. Li, Y. Wu and C.Q. Zhang 2020]

Let G be a $(3k - 3)$-edge-connected graph, where $k \geq 3$. For any $(2k, \beta)$-boundary of G, G admits a $(2k, \beta)$-orientation.

Circular $\frac{4k}{2k-1}$-flow in Eulerian signed graphs

Tutte's lemma [W.T. Tutte 1954]

If a graph admits a modulo k-flow (D, f), then it admits an integer k-flow (D, f') such that $f'(e) \equiv f(e) \pmod{k}$ for every edge e.

Lemma [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

An Eulerian signed graph (G, σ) admits a circular $\frac{4k}{2k-1}$-flow if and only if it admits a modulo $4k$-flow (D, f) such that

- for each positive edge e, $f(e) \in \{2k-1, 2k+1\}$;
- for each negative edge e, $f(e) \in \{-1, 1\}$.

Circular $\frac{4k}{2k-1}$-flow in Eulerian signed graphs

Given a signed graph (G, σ), let $d^+(v)$ denote the number of positive edges incident to v in (G, σ).

Theorem [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Given a positive integer k, an Eulerian signed graph (G, σ) admits a $\frac{4k}{2k-1}$-flow if and only if the underlying graph G admits a $(4k, \beta)$-orientation with $\beta(v) \equiv 2k \cdot d^+(v) \pmod{4k}$ for each vertex $v \in V(G)$.

Sketch of the proof

- Assume that D is a $(4k, \beta)$-orientation on G with $\beta(v) \equiv 2k \cdot d^+(v) \pmod{4k}$. Let D' be an arbitrary orientation on G.
- Define $f_1 : E(G) \to \mathbb{Z}_{4k}$ such that $f_1(e) = 1$ if e is oriented in D the same as in D' and $f_1(e) = -1$ otherwise. We claim that such a pair (D', f_1) is a modulo $4k$-flow in G satisfying that $\partial_{D'} f_1(v) \equiv \beta(v) \pmod{4k}$ for each $v \in V(G)$.
- Define $g : E(G) \to \mathbb{Z}_{4k}$ such that $g(e) = 2k$ if e is a positive edge and $g(e) = 0$ if e is a negative edge. Thus $\partial_{D'} g(v) \equiv 2k \cdot d^+(v) \pmod{4k}$ for each $v \in V(G)$.
- Let $f = f_1 + g$. Then $f : E(\hat{G}) \to \mathbb{Z}_{4k}$ satisfies the followings:
  (1) For each positive edge e, $f(e) = f_1(e) + 2k \in \{2k-1, 2k+1\}$.
  (2) For each negative edge e, $f(e) = f_1(e) \in \{-1, 1\}$.
  (3) $\partial_{D'} f(v) = \partial_{D'} f_1(v) + \partial_{D'} g(v) = \beta(v) + 2k \cdot d^+(v) \equiv 0 \pmod{4k}$.
  Such (D', f) is a circular $\frac{4k}{2k-1}$-flow in (G, σ).

Circular $\frac{4k}{2k-1}$-flow in Eulerian signed graphs

Theorem [J. Li, Y. Wu and C.Q. Zhang 2020]

Let G be a $(3k-3)$-edge-connected graph, where $k \geq 3$. For any $(2k, \beta)$-boundary of G, G admits a $(2k, \beta)$-orientation.

Theorem [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

For any Eulerian signed graph (G, σ), if the underlying graph G is $(6k-2)$-edge-connected, then $\Phi_c(G, \sigma) \leq \frac{4k}{2k-1}$.

Corollary

Every signed bipartite planar graph of girth at least $6k-2$ admits a circular $\frac{4k}{2k-1}$-coloring, i.e., it admits a homomorphism to C_{-2k}.

Bipartite folding lemma

Bipartite folding lemma [R. Naserasr, E. Rollova and E. Sopena 2013]

Let (G, σ) be a signed bipartite plane graph whose shortest negative cycle is of length $2k$. Assume that C is a facial cycle that is not of length $2k$. Then there are vertices v_{i-1}, v_i, and v_{i+1} consecutive in the cyclic order of the boundary of C, such that identifying v_{i-1} and v_{i+1}, after a possible switching at one of the two vertices, the resulting signed graph remains a signed bipartite plane graph whose shortest negative cycle is still of length $2k$.

Extending partial pre-orientation

Lemma [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Let G be a $(6k-2)$-edge-connected Eulerian graph and let z_0 be a vertex of degree $6k-2$ of G. For any $(4k,\beta)$-boundary β and its corresponding pre-orientation D_{z_0} on the edges incident to z_0 satisfying that $\overleftarrow{d}_{D_{z_0}}(z_0) - \overrightarrow{d}_{D_{z_0}}(z_0) \equiv \beta(z_0) \pmod{4k}$, D_{z_0} can be extended to a $(4k,\beta)$-orientation on G.

- Given a $(4k,\beta)$-boundary β, let D_{z_0} be the pre-orientation on the edges incident to z_0 achieving $\beta(z_0)$.
- Let D'_{z_0} be a pre-orientation obtained from D_{z_0} by changing one in-arc, say (w,z_0), of z_0 to an out-arc and let β' be defined as follows:

$$\beta'(v) = \begin{cases} \beta(v)+2 & \text{if } v = z_0, \\ \beta(v)-2, & \text{if } v = w, \\ \beta(v), & \text{otherwise.} \end{cases}$$

- We claim that D_{z_0} is extendable if and only if D'_{z_0} is extendable.

Mapping signed bipartite planar graphs to C_{-2k}

Theorem [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Every signed bipartite planar graph of negative-girth at least $6k-2$ admits a homomorphism to C_{-2k}.

Assume that (G, σ) is a minimum counterexample and (G^*, σ^*) is its dual signed graph.
By the bipartite folding lemma, we may assume that (G, σ) is a signed bipartite plane graph of negative-girth $6k-2$ in which each facial cycle is a negative $(6k-2)$-cycle and (G, σ) admits no circular $\frac{4k}{2k-1}$-coloring.

Sketch of the proof

- Assume that (X, X^c) is an edge-cut of size smaller than $6k-2$ of G^* and $|X|$ is minimized. Let $\hat{H}$ and $\hat{H}^c$ denote the signed subgraphs of $\hat{G}^*$ induced by X and X^c.
- First, $\hat{G}^*/\hat{H}$ admits a circular $\frac{4k}{2k-1}$-flow by the minimality of (G, σ). Let D be such a $(4k, \beta)$-orientation on $\hat{G}^*/\hat{H}$ with $\beta(v) \equiv 2k \cdot d^+(v) \pmod{4k}$.
- Now we consider $G_1 = \hat{G}^*/\hat{H}^c$ and we denote by z_0 the new vertex obtained by contraction.
  - Let D_{z_0} denote the orientation of D restricted on $E(z_0)$ and let β be a $(4k, \beta)$-boundary of G_1 such that $\beta(z_0) = \overleftarrow{d_{D_{z_0}}}(z_0) - \overrightarrow{d_{D_{z_0}}}(z_0)$.
  - We add $6k - 2 - d(z_0)$ many edges connecting z_0 with one vertex of $G_1 - z_0$, and orient them half toward z_0 and half away from z_0. We denote the resulting graph by G_1' and the resulting pre-orientation at z_0 by D'_{z_0}.
  - We conclude that D'_{z_0} can be extended to a $(4k, \beta)$-orientation on G_1', thus also a $(4k, \beta)$-orientation on G_1.

So the $(4k, \beta)$-orientation of $\hat{G}^*/\hat{H}$ is extended to $\hat{H}$ and thus $\hat{G}^*$ admits a $(4k, \beta)$-orientation with $\beta(v) \equiv 2p \cdot d^+(v) \pmod{4k}$.

Recent results

Circular flow index of highly edge-connected signed graphs

Edge-Conn.	Conjectured Φ_c	Known Φ_c [1]
2	$\Phi_c \leq 10$ [1]	$\Phi_c \leq 12$
3	$\Phi_c \leq 5$ [2]	$\Phi_c \leq 6$
4		$\Phi_c \leq 4$ (tight)
5	$\Phi_c \leq 3$ [3]	
6		$\Phi_c < 4$
7+planar		$\Phi_c \leq \frac{12}{5}$ [LSWW22+]
10+planar		$\Phi_c \leq \frac{16}{7}$ [LSWW22+]
$6p-2$		$\Phi_c \leq \frac{8p-2}{4p-3}$
$6p-1$		$\Phi_c \leq \frac{4p}{2p-1}$
$6p$		$\Phi_c < \frac{4p}{2p-1}$
$6p+1$		$\Phi_c \leq \frac{8p+2}{4p-1}$
$6p+2$		$\Phi_c \leq \frac{2p+1}{p}$
$6p+3$		$\Phi_c < \frac{2p+1}{p}$

[1] Almost all of the results are from [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

Conjectures

Lemma [J. Li, R. Naserasr, Z. Wang and X. Zhu 2022+]

A graph G admits a circular $\frac{p}{q}$-flow if and only if $T_2(G, +)$ admits a circular $\frac{2p}{q}$-flow.

Reformulate Tutte's 5-flow conjecture:

Conjecture [1]

Every 2-edge-connected signed graph admits a circular 10-flow.

Proposition [Z. Pan and X. Zhu 2003]

For any rational number $r \in [2, 10]$, there exists a 2-edge-connected signed graph whose circular flow index is r.

Conjectures

- Reduction of Tutte's 5-flow conjecture to 3-edge-connected cubic graphs

Conjecture [2]

Every 3-edge-connected signed graph admits a circular 5-flow.

- Stronger Tutte's 3-flow conjecture

Conjecture [3]

Every 5-edge-connected signed graph admits a circular 3-flow.

- Tutte's 4-flow conjecture restated

Conjecture

Every 2-edge-connected signed Petersen-minor-free graph admits a circular 8-flow.

Discussion

- Given an integer $k \geq 1$, what is the smallest integer $f_1(k)$ such that every $f_1(k)$-edge-connected signed graphs admits a circular $\frac{2k+1}{k}$-flow?
- Given an integer $k \geq 1$, what is the smallest integer $f_2(k)$ such that every $f_2(k)$-edge-connected signed graphs admits a circular $\frac{4k}{2k-1}$-flow?

For Eulerian signed graphs:

- Given an integer $k \geq 1$, what is the smallest integer $g(k)$ such that every (negative-)$g(k)$-edge-connected Eulerian signed graphs admits a circular $\frac{4k}{2k-1}$-flow?

Thanks for your attention!